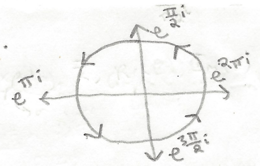
$i^n = i, -1, -i, 1, \dots$




$e^{-i\theta} \to$ clockwise, since $\overline{e^{i\theta}} = e^{-i\theta}$

$e^{i\theta} \to$ counterclockwise

$(z+w)^2 = |z| + |w| + 2\,Re(z\overline{w})$

$2(Re(z)Re(w) + Im(z)Im(w)) = 2Re(z\overline{w})$

$Re(iz) = -Im(z)$, $Im(iz) = Re(z)$

**Complex Numbers**

$z = x+iy$, $x,y \in \mathbb{R}$. Modulus: $|z| = \sqrt{x^2+y^2}$, Conjugate: $\overline{z} = x - iy \to z \in \mathbb{R} \Leftrightarrow z = \overline{z}$, $\overline{z+w} = \overline{z} + \overline{w}$, $|z|^2 = z\overline{z}$,

$\overline{zw} = \overline{z}\cdot\overline{w}$, $|zw| = |z||w|$, Polar Representation: $z = x+iy \Leftrightarrow x = |z|\cos\theta$, $y = |z|\sin\theta$, $z = |z|\,cis\,\theta$

$\arg z = \theta$ s.t. $z = |z|\,cis\,\theta$, $\operatorname{Arg} z = \theta$ s.t. $\theta = \arg z$ and $-\pi \le \theta < \pi$, $\arg zw = \theta + \phi$, $z - \overline{z} = 2\,im(z)\,i$, $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$

**properties of cis θ and z**

$cis(-\theta) = \frac{1}{cis\,\theta}$, $cis\,\theta\, cis\,\phi = cis(\theta+\phi)$, $cis\,\theta / cis\,\phi = cis(\theta - \phi)$, $zw = |z||w|\,cis(\theta+\phi)$, $\frac{z}{w} = \frac{|z|}{|w|}cis(\theta - \phi)$

De Moivre: $(cis(\theta))^n = cis(n\theta)$ Parallelogram law: $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$, law of Cosines: $|z-w|^2 = |z|^2 + |w|^2 - 2Re(z\overline{w})$

$|w||z|\cos\theta = Re(z\overline{w})$, $\theta = \arg(z-w)$, $z + \overline{z} = 2Re(z)$.

**Polynomials**

$f(x), g(x) \in \mathbb{C}[x]$, $\exists!\, h(x), r(x)$ w/ $\deg r(x) < \deg g(x)$ and $f(x) = g(x)h(x) + r(x)$. If $f(\alpha) = 0$, $\alpha \in \mathbb{C} \Rightarrow f(x) = (x-\alpha)h(x)$, $\deg h(x) = \deg f(x) - 1$

Nth root of w – $\alpha$ s.t. $\alpha^n = w$, a root of $x^n - w$, for $w = |w|\,cis\,\theta$ is $\left\{ |w|^{1/n} cis\left(\frac{\theta + 2\pi k}{n}\right)\right\}$ $0 \le k \le n$.

line btw z, w – $\{tz + (1-t)w \mid t \in \mathbb{R}\}$, $x^n - 1 = (x-1)(x^{n-1} + \dots + x + 1) = (x-1)(x-\zeta)\cdots(x - \zeta^{n-1})$

**Sequences and Series**

$\{z_n\} \to L$ $z_n \in \mathbb{C} \Leftrightarrow \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ |z_n - L| < \varepsilon$; $\forall R\ \exists N\ \forall n > N\ |z_n| > R \Rightarrow \lim_{n\to\infty} z_n = \infty$

Series: $\sum_{n=1}^{\infty} z_n$ Sequence of partial sums $\left\{\sum_{n=1}^{k} z_n\right\}_{k=1}^{\infty}$

$\log x$, $\operatorname{Arg} x$, $\arg x$, $\alpha(x)$ – Continuous except on ray.

$e^x$, $\overline{z}$, $Re(z)$, $Im(z)$, $|z|$ – Continuous on entire domain.

$\lim_{z\to w} f(z) = L$ $\forall \varepsilon > 0\ \exists \delta > 0\ |z-w| < \delta \Rightarrow |f(z) - L| < \varepsilon$. f continuous (at w) $\lim_{z\to w} f(z) = f(w)$. $f \circ g$ cont. iff f, g continuous

$z_n \to w$, f cont. at $w \Rightarrow \lim_{n\to\infty} f(z_n) = f(w)$. $\lim_{z\to w} f(z) = \infty \Leftrightarrow \forall R > 0\ \exists \delta > 0 : |z - w| < \delta$ $z \in Dom(f) \Rightarrow |f(z)| > R$

$\lim_{z\to\infty} f(z) = L \Leftrightarrow \forall \varepsilon > 0\ \exists R > 0\ |z| > R \Rightarrow |f(z) - L| < \varepsilon$ $\lim_{z\to\infty} f(z) = \infty \Leftrightarrow \forall R > 0\ \exists \delta > 0\ |z| > \delta \Rightarrow |f(z)| > R$

$f : U \to \mathbb{R}^2$ (or $\mathbb{C}$) $f = u + iv$ $u, v : U \to \mathbb{R}$ f cont $\Leftrightarrow$ u and v cont.

**Exponential function**

$e^z : \mathbb{C} \to \mathbb{C}$ $e^z = e^x cis\,y$, $z = x+iy$. $|e^z| = e^{Re(z)}$, $e^z \ne 0$, $e^{z+w} = e^z e^w$, $e^{z+2\pi i} = e^z$, $e^z = e^w \Leftrightarrow Re(z) = Re(w) \wedge$

$im(z) = im(w) + 2n\pi$, $n \in \mathbb{Z}$. $\log w = z \Leftrightarrow e^z = w$, $\log w = \ln|w| + i\arg(w)$ $Re(e^z) = e^x \cos y$, $Im(e^z) = e^x \sin y$

$z_1 = 2\pi i n + z_2 \Leftrightarrow z_1, z_2$ both $\log(w)$.

Branch – of $\arg z$ = A continuous argument function. 1) Define $U \subseteq \mathbb{C}$ 2) Define $f(x) = \theta$ some $x \in U$.

Branch logarithm Function – $= \ln|z| + \alpha(z)$, $\alpha(z)$ a branch defined above. Principal Branch: $Log(w) = \ln|w| + i\operatorname{Arg}(w)$

**Convergence**

Root test – $\lim_{n\to\infty} (c_n)^{1/n} = A$, $c_n \in \mathbb{R}^+$ Converges if $0 \le A < 1$, diverges if $A > 1$ LCT: $a_n, b_n > 0$ $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, then

Ratio test – $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = c$ Converges if $0 \le c < 1$, diverges if $c > 1$ $\sum a_n$ converges $\Leftrightarrow \sum b_n$ converges

$\sum a_n$ converges $\Rightarrow a_n \to 0$

Alternating Series test – $\sum (-1)^n b_n$ $b_n \ge 0\ \forall n$ $\lim_{n\to\infty} b_n = 0 \wedge \{b_n\}$ decreasing sequence $\Rightarrow \sum (-1)^n b_n$ Converges $[-a_1 + a_2 - a_3 + \dots]$

Divergence Test – $\sum a_n$ has $a_n \not\to 0$, $\sum a_n$ diverges. Comparison test $0 \le a_n < r_n$ $\sum r_n$ Converges $\Rightarrow \sum a_n$ Converges.

$a_n > r_n > 0$ $\sum r_n$ diverges $\Rightarrow \sum a_n$ diverges. $\sum |a_n|$ converges $\Rightarrow \sum a_n$ Converges. $\left\{\sum_{i=1}^{k} a_i\right\}_{k \in \mathbb{N}}$ Converges $\Rightarrow \sum a_i$ Converges

**Geometry**

$|z_0 - z| = r$, circle w/ centre $z_0$ and radius r. $|z - p| = \rho|z - q|$ is a circle if $\rho \ne 1$, if $\rho = 1$ it is perpendicular bisector of pq. Roots of $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\lim_{n\to\infty} \sum z_n$, $|z_n| \ge 1$ diverges unless $z_n = \{1\}$. Converges if $|z_n| < 1$.

**Powers**

$a, z \in \mathbb{C}$, $a \ne 0$, $z^a = e^{a \log z}$

**Sine / Cosine**

$e^{i\theta} = cis(\theta)$, $e^{-i\theta} = cis(-\theta) = \cos(\theta) - i\sin(\theta) \Rightarrow \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$, $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$

$\sin(-z) = -\sin(z)$, $\sin(\frac{\pi}{2} - \theta) = \cos(\theta)$, $\sin(\pi - \theta) = \sin(\theta)$, $\cos(-\theta) = \cos(\theta)$, $\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$, $\cos(\pi - \theta) = -\cos(\theta)$

$e^{z_1} = e^{z_1 + 2\pi i m}$ = any output, $\log(z) = w + 2\pi i m$ $\lim_{z\to 0} \frac{\sin(z)}{z} = 1$, $\lim_{z\to 0} \frac{1 - \cos(z)}{z} = 0$

↳ ∞ many preimages for each point in range

↳ range is $\mathbb{C}\setminus\{0\}$

↳ ∞ many images for each point in domain.

$\sum_{k=0}^{n} r^k = \frac{1-r^{k+1}}{1-r}$

$\lim_{n\to\infty} \sqrt[n]{n} = 1$

**Analytic functions**

$f$ differentiable at $z_0$ in domain $D$ if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ * Arg(h) can change arbitrarily * $|h|\to 0$

$f$ analytic if $f$ differentiable at each point in $D$. $f$ entire if $f$ analytic on $\mathbb{C}$.

polynomials - entire, $\frac{d}{dz} z^n = nz^{n-1}$, $f,g$ analytic $\Rightarrow f+g, f\cdot g$ analytic $f+g = f'+g'$, $(fg)' = fg'+f'g$

$\frac{f}{g}$ analytic if $g(z_0)\neq 0$ $\left(\frac{f}{g}\right)'(z_0) = \frac{gf'-fg'}{g^2}$, $f,g$ differentiable and range$(f)\subseteq$ Dom$(g)$ $(g\circ f)' = g'(f(z))f'(z)$

Analytic functions - rational $\left(\frac{p}{q}\right)$ if $q(x)\neq 0$, $x\in$ Dom$(p/q)$, if $f = u+iv$, $f$ analytic in $D \Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}} \to$ CRE

If $u, v, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ exist and continuous at disk centred around $z_0$ and $u,v$ satisfy CRE $\Rightarrow f$ differentiable at $z_0$.

↳ $f$ analytic: $u$ or $v$ constant $\Rightarrow f$ constant; $f$ real valued $\Rightarrow f$ constant.
$|f(z)|$ constant $\Rightarrow f$ constant; $f$ bounded, entire $\Rightarrow f$ constant (Liouville Thm)
$f(z)$ has fixed arg $\Rightarrow f$ constant
$f'(z)$ zero $\Rightarrow f$ constant

**Power Series**

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$ Coefficients: $a_i\in\mathbb{C}$, Centre: $z_0\in\mathbb{C}$.

- $f$ analytic on simply connected $U$, $\exists F$ analytic on $U$ s.t. $F'=f$

**Integrals**

- Let $m\in\mathbb{Z}$, $C$ circle centred at $0$ w/ radius $R>0$ counterclockwise. $\int_C z^m dz = \begin{cases} 0, & m\neq -1 \\ 2\pi i, & m=-1\end{cases}$ → simple if $\gamma(t)\neq\gamma(s)$ $\forall a\le t<s<b$; → $\gamma(a)=\gamma(b)$
- let $\partial\Omega = \gamma$, $f$ analytic on open set $U \supseteq \gamma\cup\Omega$, $f=u+iv$, $u_x, u_y, v_x, v_y$ continuous and $\gamma$ closed $\Rightarrow \int_\gamma f(z)dz = 0$
- $F$ analytic on open connected set $\Omega$ $f(z)=F'(z)$ continuous $\gamma:[a,b]\to\mathbb{C}$ path in $\Omega \Rightarrow \int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$
- $f:\mathbb{C}\to\mathbb{C}$ on $[a,b]$, $f=u+iv$, $u,v:\mathbb{R}\to\mathbb{R} \Rightarrow \int_a^b f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b f(\gamma(t))(x'(t)+iy'(t))dt = \dots$
- $u,v$ cont $\Rightarrow f$ cont $\Rightarrow \int_a^b f$ exists. $\left|\int_a^b f\right| \le \int_a^b |f|$, $\left|\int_\gamma f(z)dz\right| \le \max_{z\in\gamma}|f(z)| \cdot \int_a^b |\gamma'(t)|dt$ → length of $\gamma$.

inverse $= \gamma^{-1}$ → swap start and end points, go opposite direction

path $= c\in\mathbb{C}, R>0$ $\gamma:[0,2\pi]\to\mathbb{C}$ $\gamma(t) = c+Re^{it}$ $\gamma(0)=\gamma(1) = c+R$ Composition $= \gamma\alpha \to$ 1st go along $\gamma$ then $\alpha$.

Smooth path $\gamma(t) = x(t)+iy(t)$ if $x'(t), y'(t)$ exist and cont. on $[a,b]$, $\gamma'(t) = x'(t)+iy'(t)$.

$\int_\gamma f(z)dz = \int_a^b f(x,y)x'(t)dt + i\int_a^b f(x,y)y'(t)dt = \int_a^b f(x,y)dx + i\int_a^b f(x,y)dy$.

Greens thm - Let $\Omega\subseteq\mathbb{C}$ bounded, open, $\partial\Omega$ finitely many disjoint simple closed curves, $u,v$ cont. partial derivatives on open set containing $\partial\Omega\cup\Omega$. $f=u+iv$, $\Rightarrow \int_{\partial\Omega} f(z)dz = i\iint_\Omega \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\,dxdy = i\iint_\Omega [u_x+iv_x] + i[u_y+iv_y]\,dxdy$

$\int_\gamma \frac{1}{z-p}dz = \begin{cases} 0 & p\notin\Omega \\ 2\pi i & p\in\Omega\end{cases}$ $\gamma$ simple closed curve counterclockwise $\partial\Omega=\gamma$, $p\in\mathbb{C}-\gamma$.

**Radius of Convergence**

$\sum_0^\infty a_n(z-c)^n$ has ROC $=R$, If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|}$ exist $\Rightarrow$ they are equal to $\frac{1}{R}$

Abel: For every power series $\exists R\in\mathbb{R}^+\cup\{\infty\}$ s.t. $|z-c|<R \Rightarrow$ absolute convergence, $|z-c|>R\Rightarrow$ divergence.

$f(z) = \sum_0^\infty a_n(z-c)^n$ analytic on $\{z: |z-c|<R\}$, $f'(z) = \sum_0^\infty n a_n(z-c)^{n-1}$

**Simply Connected**

Simply connected - open connected set where every simple closed curve $\gamma$ in $U$ has inside in $U$.
- $\exists F$ analytic on $U$ s.t. $F'=f$
- $U$ simply connected, $f$ analytic on $U$, $\gamma$ closed polygonal curve in $U$ w/ only horizontal and vertical segments $\Rightarrow \int_\gamma f(z)dz = 0$.

**Cauchy's Formula**

- $f$ analytic on open set $U$, $\gamma$ simple closed curve in $U$ counterclockwise $(\partial\Omega=\gamma)$ $\Omega\subseteq U$, $z_0\in\Omega \Rightarrow f(z_0) = \frac{1}{2\pi i}\int \frac{f(z)}{z-z_0}dz$
- $f$ analytic on open set $U \Rightarrow \forall z_0\in U$, $R>0 \wedge B_R(z_0)\subset U \Rightarrow f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n$ $\forall z\in B_R(z_0)$. Also $a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{|\zeta-z_0|=r}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta$,

$\forall n\in\mathbb{N}$ $f^{(n)}$ exists and analytic, $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz$, $\gamma$ circle w/ radius $r<R$, centred at $z_0$.
↳ i.e. $f^{(n)}$ analytic.

Cauchy Thm - $f$ analytic on $D$, $\partial\Omega$ piecewise smooth simple closed curve, $\partial\Omega\cup\Omega\subseteq D \Rightarrow \int_{\partial\Omega} f(z)dz = 0$

$f(z) = \sum_{k=0}^\infty \underbrace{\frac{f^{(k)}(z_0)}{k!}}_{a_k}(z-z_0)^k$

**Singularities**

$z_0$ isolated singularity of $f$ if $\exists r>0$: $f$ analytic on $B_r(z_0)-\{z_0\}$, $f$ not defined at $z_0$.

1) removable: $\lim_{z\to z_0} f(z) = L$ exists $\Leftrightarrow f$ bounded near $z_0 \Leftrightarrow \tilde{f} = \begin{cases} L & z=z_0 \\ f(z), & z\neq z_0\end{cases}$ has $\tilde{f}$ analytic on $B_r(z_0)$.

2) Pole: $\lim_{z\to z_0} f(z) = \infty \Leftrightarrow \lim_{z\to z_0}(z-z_0)^k f(z)$ exists (limit DNE for $k'<k$...) ($k$ pole of $z_0$) $\Leftrightarrow f(z) = \sum_{n=-k}^\infty a_n(z-z_0)^n$, $z\in B_r(z_0)-\{z_0\}$. ← Laurent expansion near $z_0$.

↳ $\int_{|z-z_0|=r} f(z)dz = 2\pi i\, a_{-1}$ → coefficient from Laurent expansion.

3) Essential Singularity → Neither removable nor pole.

Series $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots \quad |z|<1$

$\text{Log}(1-z) = -\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \dots\right) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$

$\left(\frac{u}{v}\right)' = \frac{v\,du - u\,dv}{v^2}$

Laurent Series: $z_0$ pole of $f$, then near $z_0$ $\quad f(z) = \frac{a_{-k}}{(z-z_0)^k} + \dots + \frac{a_0}{(z-z_0)} + \sum_{n=1}^{\infty} \frac{a_n}{(z-z_0)^n}, \quad a_{-k} \neq 0.$

$k$ = smallest integer where $(z-z_0)^k f(z)$ analytic at $z_0$. * Laurent expansion is unique.

$f$ has a pole of order $k$ at $z_0 \Leftrightarrow$ near $z_0$, $f(z) = \frac{g(z)}{(z-z_0)^k}$ $g(z_0) \neq 0$.

- Radius of power series around $z_0$ is distance from $z_0$ to 1st nonremovable singularity.

→ If $U$ an open set $f: U \to \mathbb{C}$ analytic $\Rightarrow$ $f$ constant or zeros of $f$ are isolated.

$f$ has pole order 1, $\lim_{z \to z_0} (z-z_0) f(z) = \text{res}(f, z_0)$ $\quad$ $f$ has pole order $k$

$f$ has pole order 2, $\lim_{z \to z_0} g'(z),\ g(z) = (z-z_0)^2 f(z)$ $\quad$ ↳ $\text{res}(f, z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} g^{(k-1)}(z)$

↳ $g(z) = (z-z_0)^k f(z)$

→ $g$ analytic near $z_0$, $g(z_0) \neq 0$ and $h$ has pole of order $k$ at $z_0$, then $h(z)g(z)$ has pole of order $k$ at $z_0$.

If $f$ analytic on Annulus $0 \leq r < |z-z_0| < R \Rightarrow f(z) = \underbrace{\sum_{k=0}^{\infty} a_k (z-z_0)^k}_{\text{analytic on } |z-z_0|<R} + \underbrace{\sum_{k=1}^{\infty} b_k (z-z_0)^{-k}}_{\text{analytic on } r<|z-z_0|}$

$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}} dw$

↳ circle cntrd at $z_0$, $C \subseteq \Omega$

$f$ has poles $P_1, \dots, P_n$, analytic on $\mathbb{C} - \{P_0, \dots, P_n\}$ $U$ open set containing $\Omega \cup \partial\Omega$

$\Rightarrow \int_{\partial\Omega} f(z)\,dz = 2\pi i \left( \sum_{i=1}^{n} \text{res}(f, P_i) \right)$

$\text{res}(f, z_0) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = s} f(\zeta)\,d\zeta$, $f$ analytic on $0 < |\zeta - z_0| < r$, $s<r$.

↳ one pole.

$F, G$ analytic on $\{z : |z-z_0| < r_0\}$ $G(z_0) = 0$, $G'(z_0) \neq 0$, $\text{Res}\left(\frac{F}{G}; z_0\right) = \frac{F(z_0)}{G'(z_0)}$